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A NEW APPROACH TO CHARACTERIZE QUBIT CHANNELS

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We analyze qubit channels by exploiting the possibility of representing two-level quantum systems in terms of characteristic functions. To do so, we use functions of non-commuting variables (Grassmann variables), defined in terms of generalized displacement operators, following an approach which resemble the one adopted for continuous-variable (Bosonic) systems. It allows us to introduce the notion of qubit Gaussian channels and to show that they share similar properties with the corresponding continuous-variable counterpart. Some examples of qubit channels are investigated using this approach.

Keywords: Qubit channels; Grassmann algebra; Characteristic functions.

1. Introduction

In quantum information the transmission of messages through a noisy environment is described by quantum channels. They are mathematically represented by completely positive trace-preserving maps acting on the set of the density operators of the quantum system carrying information^{1,2,3,4}. For this reason the characterization and classification of such maps have attracted a lot of interest in recent years⁵. The majority of the results obtained so far relate to two specific classes of channels, i.e. the qubit channels^{6,7,8,9,10,11,12,13} and the Bosonic Gaussian channels^{14,15,16,17,18,19,20,21,22,23,24,25,26,27}.

A quantum channel \mathcal{N} can be described^{30,31,32} as a unitary interaction between the information carrier system S and an external environment E , prepared in some fixed (generally) *mixed* state ρ_E , i.e., $\mathcal{N}(\rho) = \text{Tr}_E[U(\rho \otimes \rho_E)U^\dagger]$, where $\text{Tr}_E[\dots]$ is the partial trace over the environment and U is a unitary operator in the composite Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$. Following Refs. 20, 21 we generalize the notion of *complementary channel* \mathcal{N}_{com} of \mathcal{N} of Refs. 9, 28, 29, by introducing the *weakly complementary* channel $\tilde{\mathcal{N}}$ that maps the input state ρ of the carrier into the state of the environment E after the interaction with S , i.e., $\tilde{\mathcal{N}}(\rho) = \text{Tr}_S[U(\rho \otimes \rho_E)U^\dagger]$. The channel $\tilde{\mathcal{N}}$ describes, in some sense, the quantum information lost in the en-

vironment due to the noise effects. The quantum channel \mathcal{N} is then called^{20,21} weakly degradable when, without having access to the input state of the carrier ρ , one can recover the information lost in the environment during the interaction with E only applying a local physical transformation to the final system state after the interaction. In the opposite case, the channel is called anti-degradable, i.e. the output system state can be obtained acting locally only on the environmental output state; in this case the quantum capacity can be proved to be equal to zero²⁰.

In a previous paper³³ we establish a parallelism among the qubit channels and the Bosonic Gaussian channels, through a phase-space representation of two-level quantum states³⁴. Therefore, we introduce the set of qubit Gaussian channels and we study the weak-degradability properties. Interestingly enough Gaussian maps share similar properties in both the qubit and Bosonic case. Here we briefly review these results and we apply them to some examples of qubit channels.

2. Characteristic and Green functions for qubits

The characteristic function description of a qubit can be introduced by properly adapting the formalism of Ref. 34 that generalizes the Bosonic phase space description to Fermionic systems. As shown in Ref. 33 this allows one to define the characteristic function $\chi(\xi)$ of a density matrix ρ as $\chi(\xi) \equiv \text{Tr}[\rho D(\xi)]$ where $D(\xi)$ is the qubit displacement operator $D(\xi) \equiv \exp(\sigma_+ \xi - \xi^* \sigma_-)$ with $\sigma_+ = (\sigma_-)^\dagger = |1\rangle\langle 0|$ and ξ and ξ^* being a couple of conjugate Grassmann variables³⁵. For a generic density operator of the form $\rho \equiv \begin{pmatrix} p & \gamma \\ \gamma^* & 1-p \end{pmatrix}$ this yields

$$\chi(\xi) = 1 + (2p-1) \frac{\xi \xi^*}{2} + \gamma \xi - \gamma^* \xi^*.$$

Now let us consider the action of a qubit channel \mathcal{N} on an input state represented by the density operator ρ . Using the above definitions one can write³³ the characteristic function $\chi'(\xi)$ associated with the output state $\mathcal{N}(\rho)$ as

$$\chi'(\xi) = \int d^2\zeta \chi(\zeta) G(\zeta, \xi), \quad (1)$$

where

$$G(\zeta, \xi) = \text{Tr} \left[\mathcal{N} \left(\sigma_3 D(-\zeta) \right) D(\xi) \right], \quad (2)$$

is the Green function of the map \mathcal{N} (in this expression σ_3 is the third Pauli matrix). The Green function representation (2) gives a complete description of the channel. In particular let us consider the canonical form of qubit channels of Ref. 6, i.e. $\mathcal{N}(\rho) = \mathcal{N} \left(\frac{\mathbf{1} + \vec{r} \cdot \vec{\sigma}}{2} \right) = \frac{\mathbf{1} + (\vec{t} + T \vec{r}) \cdot \vec{\sigma}}{2}$ with $\vec{t} = (t_1, t_2, t_3)$ being a real vector, $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ a vector containing the Pauli matrices, \vec{r} the Bloch vector describing the input state, and $T = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, with the real coefficients $\lambda_{1,2,3}$ and $t_{1,2,3}$

satisfying certain conditions^{6,7}. The corresponding Green function is

$$G(\zeta, \xi) = \delta^{(2)} \left(\zeta - \frac{\lambda_2 + \lambda_1}{2} \xi - \frac{\lambda_2 - \lambda_1}{2} \xi^* \right) \exp \left[-\frac{t_3}{2} \xi^* \xi \right] \\ + (\lambda_3 - \lambda_1 \lambda_2) \xi \xi^* + \frac{t_1 - it_2}{2} \zeta \zeta^* \xi - \frac{t_1 + it_2}{2} \zeta \zeta^* \xi^*, \quad (3)$$

with $\delta^{(2)}(\zeta)$ being the Dirac delta function associated with the Grassmann variable ζ ³⁵. In analogy with the Bosonic case, a qubit Gaussian channel is defined by having the Green function of the form

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - a\xi - b\xi^*) \exp[-c\xi^*\xi], \quad (4)$$

with a, b complex and c real (see Ref. 33 for details). According to Eq. (3) the Gaussian maps are obtained for $\lambda_3 = \lambda_1 \lambda_2$, $t_1 = t_2 = 0$. With a proper parametrization, the Green function of qubit Gaussian channels can thus be written as

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - \xi \cos \theta \cos \phi + \xi^* \sin \theta \sin \phi) \exp \left[(2q - 1) \frac{\cos(2\theta) - \cos(2\phi)}{4} \xi \xi^* \right]$$

with θ, ϕ in $[0, 2\pi[$ and $q \in [0, 1]$. It is worth pointing out that, analogously to what happens in the Bosonic case^{20,21}, the qubit Gaussian channels defined here can always be described as “qubit-qubit” channels, i.e. in terms of a unitary interaction between a qubit system and a *single* (not necessarily pure) qubit environment³³. Pure environment is obtained when $q = 0$ or $q = 1$, θ and ϕ generic. These channels have been proved^{13,33} to be weakly degradable for $\cos(2\theta)/\cos(2\phi) \geq 0$, and anti-degradable otherwise. If $0 < q < 1$, the single qubit environment E is initially prepared in the mixed state $\rho_E \equiv q|0\rangle_E\langle 0| + (1 - q)|1\rangle_E\langle 1|$, and the channel is weakly degradable³³ for $\cos(2\theta)/\cos(2\phi) \geq 0$ and with null quantum capacity otherwise.

3. Some qubit channels

In the following we will show the Green function of some examples of qubit quantum channels¹ and we will analyze their weak-degradability properties.

3.1. Bit flip or dephasing channel

The bit flip (or dephasing channel) flips the state $|0\rangle$ to $|1\rangle$ (and viceversa) with probability $1 - s$. This map can be obtained from the canonical form above by $t_1 = t_2 = t_3 = 0$, $\lambda_1 = 1$, and $\lambda_2 = \lambda_3 = 2s - 1$; it is a qubit-qubit map with pure environment ($q=1$). The relative Gaussian Green function is:

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - s\xi - (s - 1)\xi^*). \quad (5)$$

Observing that $\cos(2\theta) = \frac{\lambda_3 + t_3}{2} = s - 1/2$, $\cos(2\phi) = \frac{\lambda_3 - t_3}{2} = s - 1/2$, and $\cos(2\theta)/\cos(2\phi) = 1 > 0$, the bit flip channel is always weakly degradable for any value of s .

3.2. Phase flip channel

The phase flip channel changes the phase of the state $|1\rangle$ with probability $1 - s$; for instance, $\frac{1}{2}(|0\rangle + |1\rangle)$ is mapped to $\frac{1}{2}(|0\rangle - |1\rangle)$ with probability $1 - s$ and with probability s it remains unchanged. This channel has the following (canonical) parameters $t_1 = t_2 = t_3 = 0$, $\lambda_3 = 1$, and $\lambda_1 = \lambda_2 = 2s - 1$ and the relative Green function is not Gaussian, i.e.

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - (2s - 1)\xi) + 4s(1 - s)\xi\xi^* . \quad (6)$$

Since the canonical form⁷ is uniquely determined only up to unitary transformations, one can permute the λ s and so the phase flip channel is unitarily equivalent to a bit-flip channel. Therefore, the phase flip channel is not a Gaussian channel but it is unitarily equivalent to a (weakly degradable) Gaussian map.

3.3. Bit-phase flip channel

The bit-phase flip channel is a combination of a bit flip and a phase flip channels¹. Its Kraus operators are:

$$A_0 = \sqrt{s} I = \sqrt{s} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad A_1 = \sqrt{1 - s} \sigma_y = \sqrt{1 - s} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \quad (7)$$

The phase-flip channel is obtained with the following parameters $t_1 = t_2 = t_3 = 0$, $\lambda_2 = 1$, $\lambda_1 = \lambda_3 = 2s - 1$, and so it is a qubit-qubit map with pure environment (q=1). Indeed, $|t_3| = \sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)} = 0$. The Green function is so Gaussian, i.e.

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - s\xi - (1 - s)\xi^*) . \quad (8)$$

Observing that $\cos(2\theta)/\cos(2\phi) = 1 > 0$, the bit-phase flip channel is always weakly degradable for any value of s .

3.4. Depolarizing channel

The depolarizing channel represents an important kind of noise evolution, in which the qubit is *depolarized* (i.e. replaced by the completely mixed state, $I/2$) with probability s and it is left untouched with probability $1 - p$. Therefore, the output state, i.e. $\mathcal{N}(\rho) = \frac{s}{2}I + (1 - s)\rho$. In the canonical representation it is characterized by the parameters $t_1 = t_2 = t_3 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = 1 - s$, and, since the Green function is not Gaussian, i.e.

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - (1 - s)\xi) + s(1 - s)\xi\xi^* , \quad (9)$$

we are not able to discuss its degradability properties in our formalism.

3.5. Amplitude damping channel

Let us consider now a typical process of noise evolution in which a quantum system loses its energy, well implemented by the *amplitude damping* channel. In the canonical form the amplitude damping channel is given by $t_1 = t_2 = 0$, $t_3 = 1 - n$, $\lambda_1 = \lambda_2 = \sqrt{n}$, and $\lambda_3 = n$, where $1 - n$ can be thought of as the probability of losing the system energy. Since $|t_3| = \sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)} = 1 - n$, it is a qubit-qubit map with pure environment ($q=1$). The Gaussian Green function has the form

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - \sqrt{n}\xi) \exp\left[-\frac{1-n}{2}\xi^*\xi\right]. \quad (10)$$

Since $\cos(2\theta)/\cos(2\phi) = \frac{1}{2n-1}$, the amplitude damping channel is weakly degradable for $n \geq 1/2$ and anti-degradable for $n \leq 1/2$.

3.6. Generalized amplitude damping channel

Here we describe the effect of dissipation due to the presence of an external environment at finite temperature. This quantum operation, called generalized amplitude damping channel, can be described by the following Kraus operators ($s \neq 1$)¹:

$$\begin{aligned} A_0 &= \sqrt{s} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{n} \end{pmatrix}, \quad A_1 = \sqrt{s} \begin{pmatrix} 0 & \sqrt{1-n} \\ 0 & 0 \end{pmatrix}, \\ A_2 &= \sqrt{1-s} \begin{pmatrix} \sqrt{n} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \sqrt{1-s} \begin{pmatrix} 0 & 0 \\ \sqrt{1-n} & 0 \end{pmatrix}. \end{aligned} \quad (11)$$

The generalized amplitude damping channel corresponds to $t_1 = t_2 = 0$, $t_3 = (1 - n)(2s - 1)$, $\lambda_1 = \lambda_2 = \sqrt{n}$ and $\lambda_3 = n$. The Gaussian Green function is

$$G(\zeta, \xi) = \delta^{(2)}(\zeta - \sqrt{n}\xi) \exp\left[-(2s - 1)\frac{(1 - n)}{2}\xi^*\xi\right]. \quad (12)$$

It is a qubit-qubit map with mixed environment ($q \equiv s \neq 1$) and is weakly degradable for $n \geq 1/2$ and with null quantum capacity for $n \leq 1/2$.

4. Conclusions

In this work we briefly review the interesting parallelism³³ among the qubit channels and the Bosonic Gaussian channels, through a phase-space representation³⁴ in terms of generalized characteristic functions for qubits. It allows one to define the qubit Gaussian channels and to study their weak-degradability properties, showing in a elegant way a strong analogy to what is obtained for Bosonic Gaussian channels. Therefore, we use these results to describe some examples of qubit channels, which play an important role in quantum information science and its applications. We find that not all qubit channels are Gaussian but only those describable through a noisy interaction between one qubit (for the system) and one qubit (for the environment).

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